CRITICISMS AND DISCUSSIONS.

ODDLY-EVEN MAGIC SQUARES.*

A convenient classification of magic squares is found by recognition of the root as either a prime number or evenly-even, or oddly-even, or oddly-odd. These four classes have many common traits, but owing to some characteristic differences, a universal rule of construction has hitherto seemed unattainable. The oddly-even squares especially, have proved intractable to methods that are readily applicable to the other classes, and it is commonly believed that they are incapable of attaining the high degree of magical character which appears in those others.

Mr. W. S. Andrews, in *Magic Squares and Cubes,* page 183, has indeed presented a remarkable composite 10-square, a quasi 5-square, formed of twenty-five quadrate groups of consecutive terms, that is, a series of progressive 2-squares. That specimen has most of the diagonals correct, and the author ventures a prophecy that the number of such diagonals may in some way be increased. As some extensive explorations, recently made along those lines, have reached a very high latitude, the results will now be presented, showing a plan for giving to this peculiar sort, more than the ordinary magical properties.

Problem: To make oddly-even squares which shall have proper summation in all diagonal and rectangular rows except two, which two shall contain S—1 and S+1 respectively. This problem is solved by the use of auxiliary squares.

If n is an oddly-even root, and the natural series 1, 2, 3 etc. to n^2 is written in current groups of four terms, thus:

1.2.3.4.—	5.6.7.8.—	9.10.11.12.—	13.14.15.16.	etc.
0.1.2.3.—	0.1.2.3.—	0. 1. 2. 3.—	0. 1. 2. 3.	etc.
1	5	9	13	etc.

* Diagrams drawn by W. S. Andrews, Schenectady, N. Y.

then from each current group a series 0.1.2.3 may be subtracted, leaving a series 1.5.9.13 etc. to n^2—3, a regular progression of $n^2/4$ terms available for constructing a square whose side is $n/2$. As there are four such series, four such squares, exactly alike, readily made magic by well-known rules, when fitted together around a center, will constitute an oddly-even square possessing the magical character to a high degree. This will serve as the principal auxiliary. Another square of the same size must now be filled with the series 0.1.2.3 repeated $n^2/4$ times. The summation $3n/2$ being always odd, cannot be secured at once in every line, nor equally divided in the half lines, but all diagonal and all rectangular rows, except two of the latter, can be made to sum up correctly. Hence the completed square will show a minimum of imperfection.

In illustration of these general principles, a few examples will be given, beginning properly with the 2-square, smallest of all and first of the oddly-even. This is but an embryo, yet it exhibits in its nucleated cells some germs of the magical character, capable of indefinite expansion and growth, not only in connection with those of its own sort, but also with all the other sorts. Everything being reduced to lowest terms, a very general, if not a universal principle of construction may be discovered here. Proceeding strictly by rule, the series 1.2.3.4. affords only the term 1. repeated four times, and the series 0.1.2.3. taken once. The main auxiliary (Fig. 1) is a genuine quartered 2-square, equal and identical and regular and continuous every way. S=2.

1	1
1	1

Fig. 1.

0	1
2	3

Fig. 2

0	2
3	1

Fig. 3.

0	3
2	1

Fig. 4.

1	2
3	4

Fig. 5.

The second auxiliary (Fig. 2) taking the terms in direct order, has eight lines of summation, showing equality, S=3, in all four diagonals, while the four rectangular rows give inequalities 1.5 and 2.4; an exact balance of values. This second auxiliary may pass through eight reversed, inverted or revolved phases, its semimagic character being unchanged. Other orders may be employed, as shown in Figs. 3 and 4, bringing equality into horizontal or vertical rows, but not in both directions at the same time. Now any one of these variables may combine with the constant shown in Fig. 1, developing as many as twenty-four different arrangements of the 2-square, one example of which is given in Fig. 5.

It can not become magic unless all its terms are equal; a series whose common difference is reduced to zero. As already suggested, this 2-square plays an important part in the present scheme for producing larger squares, pervading them with its kaleidoscopic changes, and forming, we may say, the very warp and woof of their substance and structure.

The 6-square now claims particular attention. The main auxiliary, Fig. 6, consists of four 3-squares, each containing the series 1.5.9.13 etc. to 33. The 3-square is infantile; it has but one plan of construction; it is indeed regular and can not be otherwise, but it is imperfect. However, in this combination each of the four has a different aspect, reversed or inverted so that the inequalities of partial diagonals exactly balance. With this adjustment of subsquares the 6-square as a whole becomes a perfect quartered square, S=102; it is a quasi 2-square analogous to Fig. 1.

13	33	5	5	33	13
9	17	25	25	17	9
29	1	21	21	1	29
29	1	21	21	1	29
9	17	25	25	17	9
13	33	5	5	33	13

Fig. 6.

0	2	2	0	3	2
3	1	1	3	0	1
0	2	2	0	3	2
3	0	1	3	1	1
0	3	2	0	2	2
3	0	1	3	1	1

Fig. 7.

The four initial terms, 1.1.1.1 symmetrically placed, are now to be regarded as one group, a 2-square scattered into the four quarters; so also with the other groups 5.5.5.5 etc. Lines connecting like terms in each quarter will form squares or other rectangles, a pattern, as shown in Figure 9, with which the second auxiliary must agree. The series 0.1.2.3 is used nine times to form this second square as in Figure 7. There are two conditions: to secure in as many lines as possible the proper summation, and also an adjustment to the pattern of Fig. 6. For in order that the square which is to be produced by combination of the two auxiliaries shall contain all the terms of the original series, 1 to 36, a group 0.1.2.3 of the one must correspond with the group 1.1.1.1 of the other, so as to restore by addition the first current group 1.2.3.4. Another set 0.1.2.3 must coincide with the 5.5.5.5; another with the 9.9.9.9 and so on with all the groups. The auxiliary Fig. 7 meets these conditions. It has all diagonals cor-

rect, and also all rectangular rows, except the 2d and 5th verticals, which sum up respectively 8 and 10.

Consequently, the finished square Fig. 8 shows inequality in the corresponding rows. However, the original series has been restored, the current groups scattered according to the pattern, and although not strictly magic it has the inevitable inequality reduced to a minimum. The faulty verticals can be easily equalized by transposing the 33 and 34 or some other pair of numbers therein, but the four diagonals that pass through the pair will then become incorrect, and however these inequalities may be shifted about they can never be wholly eliminated. It is obvious that many varieties of the finished square having the same properties may be obtained by reversing or revolving either of the auxiliaries, and many more by some other arrangement of the subsquares. It will be observed that in Fig. 6 the group 21 is at the center, and that each 3-square may revolve on its main diagonal, 1 and 25, 9 and 33, 29 and 5 changing places. Now the subsquares may be placed so as to bring

13	35	7	5	36	15
12	18	26	28	17	10
29	3	23	21	4	31
32	1	22	24	2	30
9	20	27	25	19	11
16	33	6	8	34	14

Fig. 8.

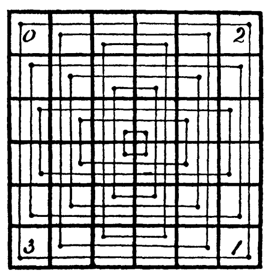

Fig. 9.

either the 5 or the 13 or the 29 group at the center, with two changes in each case. So that there may be 8×8×8=512 variations of this kind. There are other possible arrangements of the subsquares that will preserve the balance of the partial diagonals, but the pattern will be partly rhomboidal and the concentric figures tilted to right and left. These will require special adaptation of the second auxiliary.

We come now to the 10-square, no longer hampered as in the 6-square, by the imperfection of the subsquares. The main auxiliary Fig. 10 consists of four 5-squares, precisely alike, each containing the series 1.5.9 etc. to 97, S=245, in every respect regular and continuous. All four face the same way, but they might have been written right and left, as was *necessary* for the 3-square. The

groups 1.1.1.1, 5.5.5.5 etc. are analogously located, and the pattern consists of equal squares, not concentric but overlapping. The 10-square as a whole is regular and continuous. S=490.

73	29	85	41	17	73	29	85	41	17
45	1	77	33	89	45	1	77	33	89
37	93	49	5	61	37	93	49	5	61
9	65	21	97	53	9	65	21	97	53
81	57	13	69	25	81	57	13	69	25
73	29	85	41	17	73	29	85	41	17
45	1	77	33	89	45	1	77	33	89
37	93	49	5	61	37	93	49	5	61
9	65	21	97	53	9	65	21	97	53
81	57	13	69	25	81	57	13	69	25

Fig. 10.

The second auxiliary Fig. 11 is supposed to have at first the normal arrangement in the top line 0.3.0.0.3.2.2.1.2.2. which would lead to correct results in the rectangular rows, but an alternation of values in all diagonals, 14 or 16. This has been equalized

0	3	1	0	3	2	2	0	2	2
3	0	2	3	0	1	1	3	1	1
0	3	1	0	3	2	2	0	2	2
3	0	2	3	0	1	1	3	1	1
0	3	1	0	3	2	2	0	2	2
3	0	3	3	0	1	1	2	1	1
0	3	0	0	3	2	2	1	2	2
3	0	3	3	0	1	1	2	1	1
0	3	0	0	3	2	2	1	2	2
3	0	3	3	0	1	1	2	1	1

Fig. 11.

by exchange of half the middle columns, right and left, making all the diagonals = 15, but as the portions exchanged are unequal those two columns are unbalanced. The exchange of half columns might have taken place in the 1st and 8th, or in the 2d and 6th, either

the upper or the lower half, or otherwise symmetrically, the same results following.

The resultant square Fig. 12 contains all the original series, 1 to 100; it has the constant S=505 in thirty-eight out of the total of forty rows. When made magic by transposition of 15 and 16, or some other pair of numbers in those affected columns, the four diagonals that pass through such pair must bear the inequality.

73	32	86	41	20	75	31	85	43	19
48	1	79	36	89	46	2	80	34	90
37	96	50	5	64	39	95	49	7	63
12	65	23	100	53	10	66	24	98	54
81	60	14	69	28	83	59	13	71	27
76	29	88	44	17	74	30	87	42	18
45	4	77	33	92	47	3	78	35	91
40	93	52	8	61	38	94	51	6	62
9	68	21	97	56	11	67	22	99	55
84	57	16	72	25	82	58	15	70	26

Fig. 12.

Here, as in the previous example, the object is to give the second auxiliary equal summation in all diagonals at the expense of two verticals, and then to correct the corresponding error of the finished square by exchange of two numbers that differ by unity.

In all cases the main auxiliary is a quartered square, but the second auxiliary is not; hence the completed square cannot have the half lines equal, since S is always an odd number. However, there are some remarkable combinations and progressions. For instance in Fig. 12 the half lines in the top row are 252+253; in the second row 253+252; and so on, alternating all the way down. Also in the top row the alternate numbers 73+86+20+31+43=253 and the 32, 41 etc. of course = 252. The same peculiarity is found in all the rows. Figs. 10 and 11 have similar combinations. Also Figs. 6, 7 and 8. This gives rise to some Nasik progressions. Thus in Fig. 10 from upper left corner by an oblique step one cell to the right and five cells down: 73+29+85+41 etc. ten terms, practically the same as the top row = 490. This progression may be taken right or left, up or down, starting from any cell at pleasure. In Fig. 11 the ten terms will always give the constant S=15 by the

knight's move 2:1 or 1:2 or by the elongated step 3:1. Fig. 12 has not so much of the Nasik property. The oblique step one to the right and five down, 73+29+86+44 etc. ten terms = 505. This progression may start from any cell moving up and down, right and left by a sort of zigzag. The second auxiliary is richest in this Nasik property, the main auxiliary less so, as it is made by the knight's move; and the completed square still less so, as the other two neutralize each other to some extent. A vast number of variations may be obtained in the larger squares, as the subsquares admit of so many different constructive plans.

The examples already presented may serve as models for the larger sizes; these are familiar and easily handled, and they clearly show the rationale of the process. If any one wishes to traverse wider areas and to set down more numbers in rank and file, no further computations are required. The terms 0.1.2.3 are always employed: the series 1.5.9 etc. to 97, and after that 101.105.109 and so on. The principal auxiliary may be made magic by any approved process as elegant and elaborate as desired, the four subsquares being facsimiles. The second auxiliary has for all sizes an arrangement analogous to that already given which may be tabulated as follows:

6-square,	0 3 0 — 2 2 2	top row
10-square,	0 3 0 0 3 — 2 2 1 2 2	" "
14-square,	0 3 3 0 0 0 3 — 2 2 2 1 2 2 1	" "
18-square,	0 3 3 3 0 0 0 0 3 — 2 2 2 2 1 2 1 1 2	" "
etc.		

The top row being thus written, under each term is placed its complement, and all succeeding rows follow the same rule, so that the 1st, 3d, 5th etc. are the same, and the 4th, 6th, 8th etc. are repetitions of the 2d. This brings all the 0.3 terms on one side and all the 1.2 terms on the opposite. In columns there is a regular alternation of like terms; in horizontals the like terms are mostly consecutive, thus bringing the diagonals more nearly to an equality so that they may be corrected by wholesale at one operation. This systematic and somewhat mechanical arrangement insures correct summation in rows and columns, facilitates the handling of diagonals, and provides automatically for the required pattern of the 2-squares, in which both the auxiliaries and the completed square must agree. In making a square from the table it should be observed that an exchange of half columns is required, either the

upper or the lower half, preferably of the middle columns; but as we have seen in the 10-square, several other points may be found suitable for the exchange.

This plan and process for developing to so high a degree of excellence, the oddly-even squares, starting with the 2-square, and constantly employing its endless combinations, is equally applicable to the evenly-even squares. They do not need it, as there are many well-known, convenient and expeditious methods for their construction. However, in closing we will give a specimen of the 4-square, type of all that class, showing the pervading influence therein of the truly ubiquitous 2-square.

1	5	13	9
13	9	1	5
1	5	13	9
13	9	1	5

Fig. 13.

0	3	0	3
1	2	1	2
3	0	3	0
2	1	2	1

Fig. 14.

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

Fig. 15.

The primaries Figs. 13 and 14 as well as the complete square Fig. 15, singly and together fill the bill with no discount. Each is a quartered square, magic to a high degree. Each contains numerous 2-squares, four being compact in the quarters and five others overlapping. And there are many more variously scattered abroad especially in Fig. 14. While these specimens seem to conform exactly to foregoing rules they were actually made by continuous process using the knight's move 2:1 and 1:2. The pattern is rhomboidal.

In all the combinations here presented, and especially in these last specimens, the 2-square is pervasive and organic. "So we have a symmetry," as one of our philosophical writers has said—"which is astonishing, and might be deemed magical, if it were not a matter of intrinsic necessity."

D. F. SAVAGE.

NOTES ON ODDLY-EVEN MAGIC SQUARES.

The article on oddly-even squares by Mr. D. F. Savage in the current number of *The Monist* is a valuable contribution to the general literature on magic squares. Mr. Savage has not only clearly